USE OF TRANSCENDENTAL EQUATIONS IN ANALYTIC GEOMETRY.

By W. R. LONGLEY, Sheffield Scientific School.

The changes in the teaching of college mathematics during the last ten or fifteen years have been made with varying degrees of rapidity in different institutions. The more radical members of the teaching profession have abolished separate courses in trigonometry, analytic geometry, and calculus and have substituted a sequence of courses consisting of Mathematics I, Mathematics II, etc., the content of each course being a function of several variables. Among the more conservative, the classic names of courses have been retained but the content and emphasis have been more or less altered. This is particularly noticeable in analytic geometry. Some years ago an elementary course in this subject was synonymous with a thorough study of the properties of the conic sections. The tendency now is toward a very brief treatment of conics and the introduction of other material having a closer connection with problems of physics and engineering.

This movement is in line with the growing utilitarian bias of education in general and its chief impetus has come from the criticisms and demands of engineers and scientists who have use for mathematics as a tool. The teacher, trained as a pure mathematician, has yielded each point reluctantly with the feeling that the beauty and elegance of his subject are being destroyed. The beauty and elegance may possibly be preserved in some academic institutions but receive scant consideration in the technical schools, where it is felt that something has been wrong with the courses in mathematics. The engineering student has not been able to use the mathematics of the class room, and the value of the mental training received has been questioned.

The desire to improve the product of the department of mathematics, as measured by the ability of the student to use mathematics in technical and scientific work, is causing the experiments now being made. In a general way the ends to be attained are twofold. The first is the cultivation of the power of logical reasoning and the second is the ability to use mathematics as a tool. When the first end alone is sought, experience shows that the average student acquires little, if any, power to make practical applications, while a certain amount of emphasis upon the utility of topics and methods studied does not necessarily detract from their value as mental discipline. The problem before the teachers is the establishment of a proper balance.

It is to be expected that the solution of this problem will be accomplished by short steps and after a large number of trials. One step which has been taken by several writers of recent texts and which has been tried with success in the Sheffield Scientific School is the introduction of transcendental equations as a topic in analytic geometry. The problem is to solve an equation involving algebraic, trigonometric, exponential, and logarithmic expressions, for example, $1 - x = 2 \sin x$, $e^{-x} = \tan \pi x$, etc. The method is to determine the number

and approximate value of the roots graphically, for example, as the abscissas of the intersections of the curves $y = 1 - x$ and $y = 2 \sin x$. Certain roots are then calculated to a specified degree of accuracy by the use of numerical tables. The advantages of this work may be grouped under three heads.

In the first place it offers an opportunity to the teacher to test and correct the mathematical knowledge which the pupil should have acquired by previous study. The curves involved in any problem are being used as tools and, using them in this way, the student learns more of their properties than by merely studying the curves for their own sake. For some reason, difficult to explain, there is a wide difference between drawing a curve as an intermediate step in obtaining an ulterior result and in doing the exercise: "Plot the locus of $y = 4 - x^2$." The theory of logarithms, radian measure of angles, trigonometric and exponential expressions have to be *used,* and if any hazy ideas lurk in the mind of the student the teacher has no difficulty in discovering the fact.

A second advantage lies in the preparation for work to follow. For the calculus, it is a distinct help to have acquired familiarity with the behavior of trigonometric functions as *functions* (of a variable expressed in radian measure) and not merely as expressions occurring in formulas for the solution of triangles, and to have used exponential functions and logarithms to the base e. For the technical courses, the methods used here constitute an important introduction to the construction and use of engineering charts, graphical tables, and nomography.

The third and, perhaps, chief benefit lies in the use of numerical tables of different kinds with insistence upon good methods and accuracy in computation. The teacher frequently loathes numerical calculation and thinks of mathematics as the science of avoiding computation. It is of the greatest importance to consider means of avoiding unnecessary computation, but the habit of neglecting numerical work accounts for much of the dissatisfaction which outsiders feel toward the department of mathematics. Scathing criticism of the inability of college graduates to spell and to use the English language is forcing the teachers of English to give more time to fundamentals which should have been mastered in the preparatory school. The situation is much the same so far as arithmetic is concerned. A large percentage of students coming to college (with trigonometry as an entrance requirement) are not able to use numerical tables and do ordinary arithmetical calculations with speed and accuracy. Unless special attention is given to it, a two-year course in analytic geometry and calculus does not correct the deficiency. Inaccuracy in numerical work is, in general, a habit which the teachers have allowed to develop by giving too much credit for the "right method" of solving a problem. The student has come to think that it is sufficient if he "knows how" to do a certain thing and feels a rude shock when told by an employer or by an instructor in a technical course: "I care nothing about what you *know*; the only thing that counts here is what you can *do*." There is a big gap when the teacher of mathematics merely shows that it is theoretically possible to compute a certain quantity, but omits the actual computation. It is the existence of this gap that accounts, to some extent, for the

fact that so few engineers and scientists actually use their college mathematics. The closing of this gap is an important step in the improvement of the teaching of undergraduates.

In view of the importance of the methods and the value of the practice in solving transcendental equations it seems worth while to give more attention to the topic than is usually done. The texts now merely give a few such equations for solution with no indication of how they may arise. Some instances where this can be done occur in the subject matter of analytic geometry. For example the maximum and minimum points on the curve $y = x \cos x$ are given by the roots of the equation $x = \cot x$. The points of intersection of two polar curves of which one is a spiral ($\rho = a\theta$ or $\rho = e^{a\theta}$) while the equation of the other involves trigonometric functions are found by solving a transcendental equation. While this introduces nothing new theoretically it causes difficulty because the method of solution has been developed with reference to rectangular coördinates.

In college mathematics, as in high school work, a greater significance is given to any topic by the introduction of some problems having a physical meaning. The first of the following examples is a real problem which arose in some construction work and was brought to a college instructor after every member of a large force of employees in an engineering office had exhausted his knowledge of algebra in an attempt to solve it. The equation is easily deduced by elementary mathematics. The equation in the second can not be deduced without a knowledge of mechanics, but the meaning of the problem and the significance of the solutions can be explained by a simple model, which may consist merely of a book and pencil.

Problem 1. A circular arch, ACB, 14 feet long, is to be constructed with an altitude $CD = 2$ feet. Required the radius of the circle.

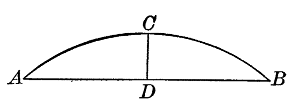

FIG. 1.

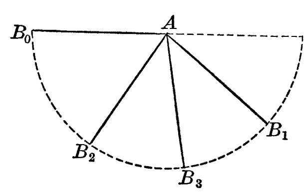

FIG. 2.

If x denotes the reciprocal of the radius, this problem leads to the equation $1 - 2x = \cos 7x$.

Problem 2.[1] A uniform rod is freely movable on a rough inclined plane, whose inclination to the horizon is i and whose coefficient of friction is μ, about a smooth pin fixed through the end A; the bar is held in the horizontal position in the plane and allowed to fall from this position. If θ be the angle through which it falls from rest show that $\sin \theta = m\theta$, where $m = \mu \cot i$.

The problem may be generalized by allowing the rod to fall from a position $\theta = \theta'$ instead of from the horizontal position $\theta = 0$. In this case, if θ increases as the rod falls, the next position of rest is given by the value of θ from the

[1] The statement of this problem is taken from Loney's *Dynamics of a Particle and of Rigid Bodies* (Cambridge, 1909), example 5, p. 217

equation (1) $\sin\theta - m\theta = \sin\theta' - m\theta'$, and, if θ decreases, θ is found from (2) $\sin\theta + m\theta = \sin\theta' + m\theta'$. Suppose the rod is allowed to fall from the horizontal position $AB_0(\theta_0 = 0)$. It swings to the position $AB_1(\theta = \theta_1)$, then back to the position $AB_2(\theta = \theta_2)$, and, after a finite number of swings, comes to rest. Successive applications of equations (1) and (2) show that the angles θ_1, θ_2, etc., are given by the abscissas of the points P_1, P_2, etc., of Fig. 3. Each segment of the zigzag line $OP_1P_2P_3$ which is directed toward the right (OP_1,

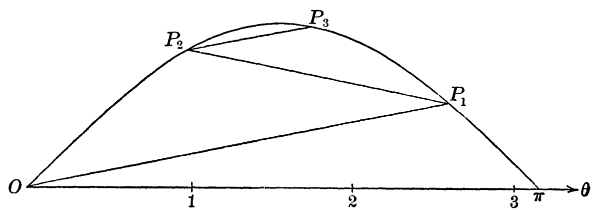

FIG. 3.

P_2P_3) has the slope m, and each segment which is directed toward the left (P_1P_2) has the slope $-m$. The figure is drawn for $i = 45°$, $\mu = 1/5$. By setting $\theta' = 0$ in equation (1) the value of θ_1 is found to be 2.595 radians. Hence the angle B_0AB_1 is equal approximately to 149°. By setting $\theta' = 2.595$ in equation (2) the value of θ_2 is found to be 0.996 radians. Hence the angle B_0AB_2 is equal approximately to 57°. By setting $\theta' = 0.996$ in equation (1) the value of θ_3 is found to be 1.733 radians. Hence the angle B_0AB_3 is equal approximately to 99°. A line drawn from P_3 with slope $-m$ will not intersect the arch of the sine curve. The force equations of the mechanical problem show that the rod will not move from this position.

Problem 3. The diameter of a bicycle wheel is 28 inches and the valve is at the lowest point of the wheel. The wheel is rolled forward until the valve is N inches ahead of its original position. Through what angle has the wheel turned? Assuming that the valve is 12 inches from the center of the wheel the equation to be solved is $N = 14\theta - 12\sin\theta$.

Problem 4. Given a string wrapped around a circle. The locus of the end as it is unwound is the involute of the circle, $x = r\cos\theta + r\theta\sin\theta$, $y = r\sin\theta - r\theta\cos\theta$. Find the length unwound when x or y have given values.

Problem 5. The equation of a damped vibration has the form $x = ae^{-bt}\sin ct$. To find the time when the moving point is at a given distance D from the center, the equation would be put in the form $De^{bt} = a\sin ct$.

A DIFFERENTIATING MACHINE.

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A differentiating machine, as its name implies, is a device for drawing the differential or rate curve of any given curve, whether the latter be a curve plotted between two variables connected by an algebraic equation or an empirical curve obtained from experimental data. Its primary interest lies in its use for

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